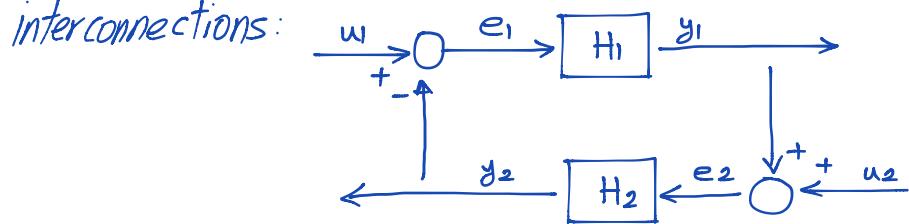


Lecture 20 (April 11, 2015)

A number of passivity theorems tell us about stability of feed back interconnections:



Theorem 6.1. The feedback connection of two passive system is passive.

$$\text{Use } V = V_1 + V_2 \rightarrow \dot{V} \leq e_1^T y_1 + e_2^T y_2 = (u - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u^T y$$

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Lemma 6.8. Feedback connection of two output strictly passive systems with

$$e_i^T y_i \geq \dot{v}_i + \delta_i y_i^T y_i, \quad \delta_i > 0$$

is finite-gain L_2 -stable and its L_2 gain $\leq \frac{1}{\delta}$ where $\delta = \min\{\delta_1, \delta_2\}$.

Proof. Let $V = V_1 + V_2$.

$$\begin{aligned} u^T y &= u_1^T y_1 + u_2^T y_2 \\ &= (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 \\ &= e_1^T y_1 + e_2^T y_2 \\ &\geq \dot{v}_1 + \dot{v}_2 + \delta_1 y_1^T y_1 + \delta_2 y_2^T y_2 \\ &\geq \dot{V} + \delta (y_1^T y_1 + y_2^T y_2) \\ &= \dot{V} + \delta y^T y \end{aligned}$$

Apply Lemma 6.5.

More general result:

Theorem 6.2. Suppose $e_i^T y_i \geq \dot{v}_i + \epsilon_i e_i^T e_i + \delta_i y_i^T y_i, i=1,2$. The closed loop map from u to y is finite gain L_2 -stable if $\epsilon_1 + \delta_1 > 0$ & $\epsilon_2 + \delta_2 > 0$.

Proof. Similar to Lemma 6.8.

Note: $e_i^T y_i \geq v_i + \varepsilon_i e_i^T e_i$ in "input strictly passive" systems.

Note: Lemma 6.8 is special case of Thm 6.2 for $\varepsilon_i = 0$ & $s_i > 0$.

Other cases:

① $s_1 = s_2 = 0$ & $\varepsilon_i > 0$ (input strictly passive)

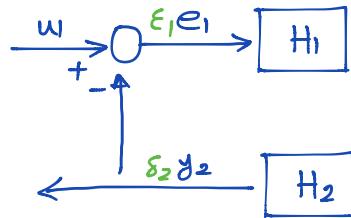
② $\varepsilon_1 = s_1 = 0$ (passive) and $\varepsilon_2 > 0$, $s_2 > 0$

more interestingly:

③ $\varepsilon_1 < 0$ & $s_2 > |\varepsilon_1|$: shortage of passivity (at the input side) of H_1
is compensated by excess of passivity (at the output side) of H_2 .

similarly:

④ $s_2 < 0$, $|s_2| < \varepsilon_1$



Example:

$$H_1: \begin{cases} \dot{x} = f(x) + G(x)e_1 \\ y_1 = h(x) \end{cases}, \quad H_2: y_2 = K e_2 \quad K > 0$$

$$e_i, y_i \in \mathbb{R}^P$$

Suppose \exists pos. def V_1 st. $\frac{\partial V_1}{\partial x} f(x) \leq 0$, $\frac{\partial V_1}{\partial x} G(x) = h^T(x) \quad \forall x \in \mathbb{R}^n$

Moreover, H_2 satisfies $e_2^T y_2 = K e_2^T e_2 = \gamma K e_2^T e_2 + \frac{1-\gamma}{K} y_2^T y_2 \quad 0 < \gamma < 1$

Therefore, the closed loop map from u to y is finite-gain L_2 stable.

We saw that H_1 is passive. Therefore, The conditions of Thm 6.2 are satisfied with $\varepsilon_2 = \gamma K$, $s_2 = \frac{1-\gamma}{K}$ and $\varepsilon_1 = s_1 = 0$.

$$e_i^T y_i \geq v_i + \varepsilon_i e_i^T e_i + s_i y_i^T y_i$$

$$\varepsilon_1 + s_2 > 0 \quad \& \quad \varepsilon_2 + s_1 > 0$$

Lyapunov stability:

Theorem 6.3. The origin of the feedback connection when $u=0$ is a.s. if

- H_1 & H_2 are strictly passive.
- H_1 & H_2 are output strictly passive & zero state observable.
- one is strictly passive and the other is output strictly passive and zero-state observable.

Furthermore, if the storage function of each component is radially unbounded, the origin is g.a.s.

Proof. Idea: Use $V = V_1 + V_2$ as a Lyapunov function candidate of the closed-loop system. We need to show that $\dot{V} \leq 0$.

Note that we only need to show that $\dot{V} \leq 0$ and not necessarily $\dot{V}_i \leq 0$. So one term, say V_i , could be negative as long as $\dot{V} \leq 0$. (again shortage of passivity of one component can get compensated by excess of passivity in other component.)

Loop transformation

